Journal of Engineering Mathematics, Vol. 4, No. 3, July 1970
Wolters-Noordhoff Publishing–Groningen
Printed in the Netherlands

An Asymptotic Theory of Guided Waves

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(Received January 8, 1970)

SUMMARY

A procedure is developed for finding asymptotic expansions at high frequencies of the solutions of Helmholtz's equation subject to boundary conditions on certain guiding surfaces. This includes surface waves along surfaces of rather general shapes, and wave-guide modes in a class of non-uniform waveguides. Guided waves have some features of both eigenvalue (mode) and radiation problems. The method of this paper combines the two techniques, finding "modes" that propagate along rays in the general waveguide region and whose amplitudes vary along the paths of propagation. The phases of these modes are found from two coupled equations, one analogous to the eiconal equation of geometrical optics, and the other analogous to the eigenvalue or "transverse resonance" equation of waveguides. The amplitudes are asymptotic series in inverse powers of the wavenumber, and the coefficients satisfy a set of ordinary differential equations that can be solved recursively. It is found that the ray paths are not only functions of the refractive index (as in "pure" radiation problems), but depend also on the local geometrical properties of the guiding surface.

1. Introduction

We consider asymptotic solutions of the reduced wave equation

$$[\nabla^2 + k^2 n^2(X)]\, U(X) = 0, \qquad X \in R \tag{1}$$

where k is a large parameter, $n(X) \in C^1$, and R is a region in a two- or three-dimensional Euclidean space.

The nature of the solutions of (1) depends of course on the properties of R, and the boundary conditions to be satisfied by U on ∂R, the boundary surface of R.

When R is the entire space or the exterior of a convex region, we have a radiation or diffraction problem. A solution of (1) exists for all k. Asymptotic solutions for large k, based on the ray method, have been constructed for a broad class of such problems. Keller and Lewis [1] describe the technique and give a bibliography that summarizes the state of the art up to 1964. A book by the same authors [2] containing many more results, will be published soon.

A ray solution is a local solution, i.e. it does not depend on the properties of R, ∂R and $n(X)$ everywhere in $R \cup \partial R$, but only on $n(X)$ along certain curves, the rays, which are the characteristics of an auxiliary equation to (1). The ray method is a powerful analytic approximation method. It can be applied to (1) when a rigorous solution (by separation of variables) cannot be found, and quite often it even leads more directly to a good approximate solution in cases that separation of variables is possible.

There exists however a class of problems in which the number of rays that pass through every observation point is large, or even infinite. For example, if R is a convex or bounded region (a "cavity"), the rays are reflected at the boundaries, and all multiple reflections have to be taken into account. A modified ray method for the asymptotic solution of such problems has been developed by Keller and Rubinow [3]. It is applicable when the characteristic dimensions of R are large in comparison to the wavelength λ, where

$$\lambda = 2\pi/k\,. \tag{2}$$

Solutions exist only for certain values of k, the eigenvalues. The solution associated with an eigenvalue is an eigenfunction or a mode of R. Guided wave problems have some features of radiation problems and some features of eigenvalue problems. Both ray and mode techniques have been applied in their solution. Usually, when the number of modes that can propagate in a structure is large, a ray approach may be preferable [4], [5]. When the number of propagating modes is small, as for example in wave guides whose cross-section dimensions are of the same order of magnitude as the wavelength, the ray method loses accuracy and the mode approach could be preferable. However, the modes are eigenfunctions of the structure. There is no rigorous way of finding the modes of a structure if it is non-uniform, i.e. when some separation of variables approach cannot be applied.

In this paper we shall use a combination of mode and ray method to analyze a class of guided wave problems. This method has been used first by J. B. Keller [6] for analyzing surface waves in water of variable depth. It consists of making an "Ansatz" for an asymptotic expansion of the solution of (1) and getting a recursive system of equations for the terms in the expansion, as is done in the ray method. The Ansatz takes into consideration the modal structure of the field. Thus, the asymptotic solution consists of a set of "modes" that propagate along two dimensional rays along a guiding surface. These modes resemble locally the modes in a uniform guiding structure. The rays are determined by an "equivalent refractive index" that combines the real refractive index n and functions of the local shape of the guiding structure. The amplitude of each mode is assumed to be an asymptotic series in k^{-n} and it varies along the path of propagation.

Many physical problems in acoustics, electromagnetism, elastodynamics [7], quantum mechanics etc., can be analysed by this method. We shall illustrate the method with the following examples: a) Scalar surface waves guided by a single surface, b) Scalar waves in a non-uniform three dimensional waveguide, c) Scalar waves in a two dimensional non-uniform, curved waveguide. In the appendix we summarize those assumptions and results of the ray method which are pertinent to our problem.

2. Waves Guided by a Single Surface (Surface Waves)

We shall look for solution of eq. (1), where

$$\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial Y^2 + \partial^2/\partial z^2 \tag{3}$$

and R is defined by

$$Y \geqq H(x, z) . \tag{4}$$

We shall assume that n does not depend on Y, i.e. $n = n(x, z)$. The boundary of R is the given surface $Y = H(x, z)$ where H is a nonnegative, bounded and differentiable function of x and z. $U(\boldsymbol{X})$ satisfies the "impedance" boundary condition

$$\frac{\partial U}{\partial v} + kZU = 0 \quad \text{on} \quad Y = H \tag{5a}$$

where $\partial/\partial v$ stands for differentiation in the direction of the normal to H and $Z = Z(x, z)$ is a given, continuous function. Eq. (5a) can be rewritten as

$$[U_Y - \nabla_t H \cdot \nabla_t U) + kZU[1 + (\nabla_t H)^2]^{\frac{1}{2}} = 0 \quad \text{on} \quad Y = H \tag{5b}$$

where $U_Y = \partial U/\partial Y$, $\nabla_t = (\partial/\partial x, \partial/\partial z)$.

We now introduce the following change of variables:

$$kY = y; \quad kH = h; \quad U(x, Y, z) = u(x, y, z) . \tag{6}$$

Under this change of variables our problem becomes

$$Lu = (\nabla_t^2 + k^2 \partial^2/\partial y^2 + k^2 n^2) u = 0 \quad \text{in} \quad y \geqq h , \tag{7}$$

$$k^2 [u_y + ZDu] - \nabla_t h \cdot \nabla_t u = 0 \qquad \text{on} \quad y = h , \tag{8}$$

where $D = D(x, z) = [1 + (\nabla_t H)^2]^{\frac{1}{2}}$. (When the slopes of H are gentle $D \simeq 1$, and (8) is further simplified). In addition to (7) and (8) the function u must tend to zero as $y \to \infty$ if it is a wave guided by the surface $y=h$, i.e.

$$\lim_{y\to\infty} |u(x, y, z)| = 0 . \tag{9}$$

We may assume now without loss of generality that the solution u of the boundary value problem eqs. (7), (8) and (9) has the form

$$u = A \exp[ik\sigma + \gamma(y-h)] \tag{10}$$

where $A = A(x, y, z)$; $\gamma = \gamma(x, z)$; $\sigma = \sigma(x, z)$. We delete henceforth the subscript t from ∇_t and denote for shortness $\gamma(y-h) = \phi$. Thus

$$\nabla u = [ik\nabla\sigma A\, e^{\phi} + \nabla(A\, e^{\phi})]\, e^{ik\sigma}$$

$$\nabla^2 u = [-k^2(\nabla\sigma)^2 A e^{\phi} + 2ik\nabla\sigma\cdot\nabla(A\, e^{\phi}) + ik\nabla^2\sigma(A\, e^{\phi}) + \nabla^2(Ae^{\phi})]\, e^{ik\sigma}$$

$$\partial u/\partial y \equiv u_y = (A_y + A\gamma)\, e^{\phi + ik\sigma}$$

$$\partial^2 u/\partial y^2 \equiv u_{yy} = (A_{yy} + 2A_y\gamma + A\gamma^2)\, e^{\phi + ik\sigma} .$$

The boundary value problem becomes:

$$Lu = e^{ik\sigma}[k^2\{[n^2 - (\nabla\sigma)^2 + \gamma^2] A\, e^{\phi} + e^{-\phi}\partial/\partial y(A_y\, e^{2\phi})\} + + ik[2\nabla\sigma\cdot\nabla(A\, e^{\phi}) + \nabla^2\sigma A\, e^{\phi}] + \nabla^2(A\, e^{\phi})] = 0 \tag{11}$$

$$k^2[A_y + (\gamma + ZD)A] - ik\nabla\sigma\cdot\nabla h A - \nabla h\cdot\nabla(A\, e^{\phi}) = 0 \text{ at } y = h . \tag{12}$$

We assume now that A in (10) has the following asymptotic expansion:

$$A(x, y, z) \sim A_0(x, z) + \sum_{m=1}^{\infty} A_m(x, y, z)(ik)^{-m} . \tag{13}$$

We substitute this expansion in (11) and (12), and equate separately to zero the coefficients of each power of k. Equating to zero the coefficient of k^2 in (11) and (12) yields respectively

$$(\nabla\sigma)^2 = n^2 + \gamma^2 = N^2 \tag{14}$$

$$\gamma = -ZD \tag{15}$$

In order to satisfy condition (9), the function Z (which can be complex-valued) must have a positive real part. We recognize eq. (14) as the eiconal equation, where $N = (n^2 + Z^2 D^2)^{\frac{1}{2}}$ is an "equivalent refractive index" which combines the real refractive index n, the surface impedance Z and the geometric properties of the surface H.

The solution of eq. (14) is standard [1] (see appendix). The equations of the rays $x = x(s)$, $z = z(s)$ are found from the ray equations

$$\frac{d}{ds}\left(N\frac{dx}{ds}\right) = \frac{\partial N}{\partial x}, \tag{16a}$$

$$\frac{d}{ds}\left(N\frac{dz}{ds}\right) = \frac{\partial N}{\partial z}, \tag{16b}$$

where s is an arclength parameter along the ray. Now $\sigma(x, z)$ is given by

$$\sigma[x(s), z(s)] = \sigma(s_0) + \int_{s_0}^{s} N(s')ds' , \tag{17}$$

where the integration path runs along a ray.

Next we equate to zero the coefficient of k^1 in (12), obtaining

$$A_{1y} = -\nabla h\cdot\nabla\sigma A_0 \text{ on } y = h . \tag{18}$$

The coefficient of k^1 in eq. (11) yields

$$\partial/\partial y(A_{1y}\,e^{2\phi}) = [2\nabla\sigma\cdot\nabla(A_0\,e^{\phi}) + \nabla^2\sigma\,A_0 e^{\phi}]\,e^{\phi} = = (2\nabla\sigma\cdot\nabla A_0 + \nabla^2\sigma A_0 + A_0\nabla\sigma\cdot\nabla)\,e^{2\phi}\,. \tag{19}$$

We can integrate eq. (19) from $y=h$ to $y=\infty$. We use condition (9) and eq. (18) and get

$$-\nabla\sigma\cdot\nabla h = 2\nabla\sigma\cdot\frac{\nabla A_0}{A_0} + \nabla^2\sigma + \nabla\sigma\cdot\nabla(1/2\gamma)\,. \tag{20}$$

Eq. (20) is an ordinary differential equation along a ray. It can be shown (see eqs. (A.9) and (A.11)) that

$$\nabla\sigma\cdot\nabla \equiv N\frac{d}{ds}\,,\quad \tfrac{1}{2}\nabla^2\sigma = N\frac{d}{ds}\ln\,(N\,\delta a)^{\frac{1}{2}}\,.$$

where δa is the cross-section of an infinitesimal "tube of rays". Thus eq. (20) can be integrated, from some reference point s_0 to a point of observation s, yielding

$$A_0(s) = A_0(s_0)\left[\frac{N(s_0)\,\delta a(s_0)}{N(s)\,\delta a(s)}\right]^{\frac{1}{2}}\exp\left\{\frac{k}{2}\,[H(s)-H(s_0)] + \tfrac{1}{4}\left[\frac{1}{Z(s)\,D(s)} - \frac{1}{Z(s_0)\,D(s_0)}\right]\right\}. \tag{21}$$

Eq. (21) gives the variation of A_0 along a ray path. It is easily seen that for $H=\text{const}$, $Z=\text{const}$ we get from (21) the well known relation of geometrical optics

$$A_0(N\,\delta a)^{\frac{1}{2}} = \text{constant along a ray.}$$

The "ray tube cross-section ratio" $\delta a(s)/\delta a(s_0)$ can be shown [1], [2], to be the Jacobian of the transformation from the xz coordinates to the new coordinate system of rays and wavefronts. To sum up: if σ is given on some initial manifold such as a point or a curve on the surface $Y=H(x, z)$, we can find the rays *via* eqs. (16a, 16b) and the function $\sigma(x, z)$ *via* eq. (17). The rays comprise of a family of curves along the guiding surface. Once the rays are found, the variation of the expansion coefficient $A_0(x, y)$ along a ray can be found *via* eq. (21). If A_0 is also given on some initial manifold the first term in the expansion of U can be found everywhere. We can find, in principle, all the expansion coefficients A_n ($n=1, 2, 3, \ldots$) by equating to zero the coefficients of k^{-m} ($m=0, 1, 2, \ldots$) in eqs. (11) and (12), similar to the procedure we used above. The end result, collected from eqs. (6), (10), (14), (15), (17) and (21) is:

$$U(x, Y, z) = \exp\{ik\sigma - k(Y-H)Z[1+(\nabla_t H)^2]^{\frac{1}{2}}\}\{A_0(x, z)[1+O(k^{-1})]\}\,. \tag{22}$$

Appropriate variations of H and Z can have focusing effects on the two-dimensional rays, causing them to converge towards caustics or foci. As usual, this technique breaks down in the neighborhood of caustics and foci, where $\delta a\to 0$. Asymptotic expansions that are uniform in the vicinity of caustics [8] can be used to overcome such difficulties.

If the surface $Y=H$ or the impedance Z has curves along which they are discontinuous, there will occur reflection and transmission phenomena when a surface wave is incident upon such curves. Since we reduced the surface wave problem to a two-dimensional radiation problem, such reflection and transmission phenomena can be calculated. This however, requires some additional analysis which we are presently investigating.

If the problem is two-dimensional, i.e. independent of the z variable, the guiding surface is $Y=H(x)$. The ray path is the x-axis and the problem simplifies considerably [7].

3. Waves Guided Between Two Surfaces (Waveguide Modes)

When the region R in eqs. (1) and (3) is given by

$$H(x, z) \geqq Y \geqq -H(x, z) \tag{23a}$$

with the boundary conditions

$$\frac{\partial U}{\partial v} \pm kZU = 0 \text{ on } Y = \pm H(x) \tag{23b}$$

we have a waveguide problem at hand. The assumptions on H and Z that were made in the former section are made here as well. We shall use the change of variables (6) and obtain

$$Lu = (\nabla_t^2 + k^2\, \partial^2/\partial y^2 + k^2 n^2)u = 0 \qquad |y| \leqq h \tag{24}$$

$$k^2[u_y \pm ZDu] \mp \nabla_t h \cdot \nabla_t u = 0\,. \qquad y = \pm h \tag{25}$$

Solutions to the boundary value problem posed by eqs. (24), (25) can be symmetrical or antisymmetrical with respect to the plane $y=0$. Thus we will assume (in analogy to eq. (10))

$$u_s = A \cos \gamma y\, e^{ik\sigma} \tag{26a}$$

$$u_a = B \sin \gamma y\, e^{ik\sigma} \tag{26b}$$

Substituting (26a) into (24) and (25) and denoting $\phi = \gamma y$, we get

$$Lu = e^{ik\sigma}\left[\left\{k^2[n^2 - (\nabla\sigma)^2 - \gamma^2]\, A \cos\phi + \frac{1}{\cos\phi}\frac{\partial}{\partial y}(A_y \cos^2\phi)\right\} + \right.$$
$$\left. + ik[2\nabla\sigma\cdot\nabla(A\cos\phi) + \nabla^2\sigma\, A\cos\phi] + \nabla^2(A\cos\phi)\right] = 0 \tag{27}$$

$$k^2[A_y \cos\phi - \gamma A \sin\phi \pm ZDA\cos\phi] \mp ik\,\nabla\sigma\cdot\nabla h\, A\cos\phi \mp$$
$$\mp \nabla h\cdot\nabla(A\cos\phi) = 0 \text{ at } y = \pm h\,. \tag{28}$$

We make now the same assumption as in eq. (13) and follow the same procedure. Equating to zero the coefficient of k^2 in eq. (27) yields

$$(\nabla\sigma)^2 = n^2 - \gamma^2 = N^2 \tag{29}$$

while from eq. (28) we get

$$(\gamma h)\tan(\gamma h) = DZh\,. \tag{30}$$

Eq. (30) is a trancendental equation for the determination of γ: Since Z, D and h are given functions of x and z, we can determine from it the set of eigenvalues $\gamma_j = \gamma_j(x, z)$ $(j = 1, 2, 3, \ldots)$. Once the eigenvalues are found, they are substituted back into eq. (29), yielding a "modal eiconal equation". The "equivalent refractive index" N is different for each mode. We can easily see that N^2 is positive only for a finite set of eigenvalues $\gamma_j (j = 1, 2, \ldots m)$. These are the propagating modes. All other modes will yield an imaginary σ and correspond to evanescent modes.

If we substitute (26b) in (27) and (28), we get in a similar way eq. (29), but eq. (30) will be replaced by

$$(\gamma h)\cot(\gamma h) = -DZh \tag{31}$$

for the antisymmetrical modes.

The solution of eq. (29) follows as usual, and yields the ray trajectories and the function $\sigma_j(x, z)$. This was described in the previous section.

The equations for the determination of the coefficients A_m are obtained as before, by equating to zero the coefficients of k^n $(n = 1, 0, -1, 2, \ldots)$. If we equate to zero the coefficient of k^1 in eqs. (28) and (27) we get:

$$A_{1y} = \mp A_0 \nabla\sigma\cdot\nabla h \text{ on } y = \pm h \tag{32}$$

$$\partial/\partial y(A_{1y}\cos^2\phi) = (2\nabla\sigma\cdot\nabla A_0 + \nabla^2\sigma\, A_0 + A_0\nabla\sigma\cdot\nabla)\cos^2\phi\,. \tag{33}$$

Since the solution u_s is assumed to be symmetrical, we have

$$A_{ny} = 0 \text{ on } y = 0 \quad \text{for} \quad n = 1, 2, 3, \ldots\,. \tag{34}$$

We integrate eq. (33) from $y=0$ to $y=h$ (or $y=-h$), using eq. (34):

$$2A_{1_y}\cos^2\gamma h = (2\nabla\sigma\cdot\nabla A_0 + \nabla^2\sigma A_0 + A_0\nabla\sigma\cdot\nabla)(h+\tfrac{1}{2}\gamma^{-1}\sin 2\gamma h)\,. \tag{35}$$

Using eq. (32) and rearranging we get

$$-\nabla\sigma\cdot\nabla h\frac{\cos^2\gamma h}{h+\frac{1}{2}\gamma^{-1}\sin 2\gamma h} = \frac{\nabla\sigma\cdot\nabla A_0}{A_0} + \tfrac{1}{2}\nabla^2\sigma + \tfrac{1}{2}\frac{\nabla\sigma\cdot\nabla(h+\frac{1}{2}\gamma^{-1}\sin 2\gamma h)}{h+\frac{1}{2}\gamma^{-1}\sin 2\gamma h}\,. \tag{36}$$

This is an ordinary differential equation for A_0 along a ray. Its integration is immediate (see previous section, eqs. (20), (21)), and yields:

$$A_0(s) = A_0(s_0)\left\{N\delta a\left[H\left(1+\frac{\sin 2k\gamma H}{2k\gamma H}\right)\right]\right\}_{s_0}^{\frac{1}{2}}\times$$
$$\left\{N\delta a\left[H\left(1+\frac{\sin 2k\gamma H}{2k\gamma H}\right)\right]\right\}_s^{-\frac{1}{2}}\left\{\exp\ \frac{\cos^2 k\gamma H}{1+\sin 2k\gamma H/2k\gamma H}\bigg|_s - \frac{\cos^2 k\gamma H}{1+\sin 2k\gamma H/2k\gamma H}\bigg|_{s_0}\right\} \tag{37a}$$

Eq. (37a) gives the variation of A_0 along a ray path. The field of a symmetrical mode has the form

$$U(x, Y, z) = \exp(ik\sigma)\cos(k\gamma H)A_0(x, y)[1+O(k^{-1})]\,. \tag{37b}$$

As usual, our analysis fails near caustics i.e. where $\delta a\to 0$, and also near the cutoff of a mode $(N\to 0)$. The calculation of B_0 (for the antisymmetrical modes) follows the same lines. Instead of eq. (34) we have the condition $\sin\gamma y=0$ for $y=0$. The higher order coefficients A_n, B_n $(n=1, 2, \ldots)$ are obtained recursively, by equating to zero the coefficients of $k^0, k^{-1}, \ldots$ in eqs. (27), (28).

4. The Duct Waveguide

Another example of guided waves which can be handled by the same technique is that of duct propagation. A duct will be, as before, the region R_1 defined by $-H\leqq Y\leqq H$, and the boundary ∂R_1 are the surfaces $Y=\pm H(x, z)$. However, the boundary value problem to be solved will be now

$$[\nabla^2 + k^2 n_1^2(X)]\,U_1 = 0 \qquad -H(x, z)\leqq Y\leqq H(x, z) \tag{38a}$$

$$[\nabla^2 + k^2 n_2^2(X)]\,U_2 = 0 \qquad |Y| > H(x, z) \tag{38b}$$

$$U_1 = aU_2 \qquad Y=\pm H(x, z) \tag{38c}$$

$$\frac{\partial U_1}{\partial v} = b\frac{\partial U_2}{\partial v} \qquad Y=\pm H(x, z) \tag{38d}$$

where $a=a(x, z)$ and $b=b(x, z)$ are given functions.

An additional condition for being a duct is

$$n_1(X) > n_2(X)\,. \qquad \forall X\in\partial R_1\,. \tag{39}$$

The problem is brought via the change of variables (6) to the form

$$[\nabla_t^2 + k^2(\partial^2/\partial y^2 + n_1^2)]\,u_1 = 0 \qquad -h\leqq y\leqq h \tag{40}$$

$$[\nabla_t^2 + k^2(\partial^2/\partial y^2 + n_2^2)]\,u_2 = 0 \qquad |y| > h \tag{41}$$

$$u_1 = au_2 \qquad y=\pm h \tag{42}$$

$$(k^2 u_{1_y} - \nabla_t h\cdot\nabla_t u_1) = b(k^2 u_{2_y} - \nabla_t h\cdot\nabla_t u_2) \qquad y=\pm h \tag{43}$$

There exist symmetrical and antisymmetrical solutions to that problem. For the symmetrical solution we make the Ansatz

$$u_1^{(s)} = A_1\cos\gamma_1 y\, e^{ik\sigma} \tag{44a}$$

$$u_2^{(s)} = A_2\exp[\gamma_2(h\mp y) + ik\sigma]\,. \tag{44b}$$

For the antisymmetrical solution we make the Ansatz

$$u_1^{(a)} = B_1 \sin \gamma_1 y\, e^{ik\sigma}\,, \tag{45a}$$

$$u_2^{(a)} = \pm B_2 \exp\left[\gamma_2 (h \mp y) + ik\sigma\right], \tag{45b}$$

where the upper and lower signs apply for $y \gtrless h$ respectively. Substituting (44a, 44b) in eqs. (40), (41) and equating to zero the coefficient of the highest power of k as before, yields

$$(\nabla\sigma)^2 = n_1^2 - \gamma_1^2 = n_2^2 + \gamma_2^2\,. \tag{46}$$

From eqs. (42), (43) and (46) we get for the symmetrical solution

$$(\gamma_1 h) \tan(\gamma_1 h) = (b/a)\gamma_2 h = b/a\left[h^2(n_1^2 - n_2^2) - (\gamma_1 h)^2\right]^{\frac{1}{2}} \tag{47}$$

while for the antisymmetrical solution we get

$$(\gamma_1 h) \cot(\gamma_1 h) = -(b/a)\gamma_2 h = -b/a\left[h^2(n_1^2 - n_2^2) - (\gamma_1 h)^2\right]^{\frac{1}{2}}\,. \tag{48}$$

Equations (46) and (47) or (48) are sufficient for the determination of γ_1, γ_2 and σ for each mode. It is worth mentioning that eq. (47) yields a real γ_1 in the range $0 \leqq \gamma_1 \leqq (n_1^2 - n_2^2)^{\frac{1}{2}}$ for any $h \geqq 0$. Thus, there exists a symmetrical mode that has no cutoff. We note that eqs. (47) and (48) correspond to eqs. (30) and (31), with ZD replaced by $b/a\left[(n_1^2 - n_2^2) - \gamma_1^2\right]^{\frac{1}{2}}$. This implies that the effect of the spaces $|y| > h$ on the solution u_1 in the region $|y| < h$ is equivalent to a non constant surface impedance on $y = \pm h$. This fact has already been observed in the analysis of diffraction at a curved interface between two media [9]. The coefficients A_1, and B_1 in eqs. (44) and (45) are assumed to have asymptotic expansions like A in eq. (13), and the calculation of the coefficients A_{1n}, B_{1n} ($n=0, 1, 2, \ldots$) proceeds as in the previous section (eqs. (32) through (37b)). The coefficients A_{2n}, B_{2n} of U_2 (eq. (38b)) are obtained from the boundary condition eq. (38c).

5. Other Guiding Structures

We could regard the "wave guides" of the two previous sections as "plane" wave guides in the following sense: the guiding surface $Y = H$ was given in terms of its distance from the "ground plane" $Y = 0$. Accordingly, the Ansatz we chose for the solution was of the form

$$A(x, y, z) \sin\left[\gamma(x, z) y\right] \exp\left[ik\sigma(x, z)\right]$$

or

$$A(x, y, z) \exp\left[\gamma(x, z)(y - h) + ik\sigma(x, z)\right].$$

The approximate solution is given in the geometrical terms H, D and dH/ds, which measure the distance of H and its inclination with respect to the "ground plane" $Y = 0$. This is by no means the only possible way. We shall give an example in which the radius of curvature of H will appear in the approximate solution. In order to avoid unnecessary tedious calculations, we shall choose a two dimensional problem: surface waves guided by a cylindrical surface (given in polar coordinates (R, θ) by $R = A + H(\theta)$, (A = constant) with an impedance boundary condition. The problem to be solved is

$$(\nabla^2 + k^2 n^2) U = \left(\frac{\partial^2}{\partial R^2} + \frac{1}{R}\frac{\partial}{\partial R} + \frac{1}{R^2}\frac{\partial^2}{\partial \theta^2} + k^2 n^2\right) U = 0 \qquad 0 \leqq R \leqq A + H(\theta) \tag{49}$$

$$\frac{\partial U}{\partial v} + kZU = 0 \text{ at } R = A + H(\theta)\,. \tag{50}$$

An additional condition is that U be regular at $R = 0$. Z and n may be functions of θ (or of $s = A\theta$).

We introduce the following change of variables

$$kR = r;\; kA = a;\; kH = h;\; \frac{\partial}{\partial\theta} = A\frac{\partial}{\partial s} = \frac{a}{k}\frac{\partial}{\partial s};\; U(R, \theta) = u(r, s)\,. \tag{51}$$

Equations (49) and (50) become

$$k^2(\partial^2 u/\partial r^2 + 1/r\ \partial u/\partial r + n^2\ u) + a^2/r^2\ \partial^2 u/\partial s^2 = 0 \tag{52}$$

$$k^2(\partial u/\partial r + ZDu) - a^2/r^2\ \partial u/\partial s \cdot dh/ds = 0 \tag{53}$$

where

$$D = [1 + (A/R\ dH/ds)^2]^{\frac{1}{2}}\ . \tag{54}$$

We now make the Ansatz

$$u(r, s) = B(r, s)\, J_\nu(nr)\, e^{ik\sigma(s)}\ . \tag{55}$$

We denote $\partial/\partial s\, f(s) = f_s = f'$; $J_\nu(nr) = J$ and $d/dx\ J_\nu(x) = \dot{J}$. Thus we obtain

$$u_s = [ik\sigma' BJ + (BJ)'] e^{ik\sigma}$$

$$u_{ss} = [-k^2\sigma'^2 BJ + 2ik\sigma'(BJ) + ik\sigma'' BJ + (BJ)''] e^{ik\sigma}$$

$$u_r = (B_r J + Bn\dot{J}) e^{ik\sigma}$$

$$u_{rr} = (B_{rr} J + 2nB_r \dot{J} + Bn^2 \ddot{J}) e^{ik\sigma}\ .$$

Substituting in eqs. (52) and (53) we get

$$[k^2\{n^2 B(\ddot{J} + 1/nr\,\dot{J} + (1 - \sigma'^2 a^2/n^2 r^2)J + 1/J\ \partial/\partial r(B_r J^2) + 1/r\ B_r J\} + $$
$$+ ik[2a^2/r^2\ \sigma'(BJ)' + a^2/r^2\ \sigma'' BJ] + a^2/r^2\ (BJ)''] e^{ik\sigma} = 0 \tag{56}$$

$$[k^2(Bn\dot{J} + ZDBJ + B_r J) - ik\sigma' h' a^2/r^2\ BJ - (BJ)' h' a^2/r^2] e^{ik\sigma} = 0 \tag{57}$$

We assume that B has the following asymptotic expansion:

$$B(r, s) \sim B_0(s) + \sum_{m=1}^{\infty} B_m(r, s)(ik)^{-m}. \tag{58}$$

We substitute (58) into (56) and (57) and equate to zero the coefficient of each power of k. The coefficient of k^2 yields

$$\ddot{J} + 1/nr\ \dot{J} + (1 - \sigma'^2 a^2/n^2 r^2)J = 0 \tag{59}$$

$$n\dot{J} + ZDJ = n\dot{J}_\nu(kn\ R) + ZDJ_\nu(kn\ R) = 0 \text{ at } R = A + H\ . \tag{60}$$

Equation (59) is an identity if we let

$$\sigma'^2 = \nu^2/a^2 \tag{61a}$$

which implies

$$\sigma(s) = \sigma(s_0) + \int_{s_0}^{s} \nu/kA\, ds\ . \tag{61b}$$

Eq. (61a) is the equivalent of the eiconal equation. We have an immediate solution to the eiconal equation, since in two dimensions the problem is trivial and the ray trajectories are the guiding circle $R = A$. The yet unknown function $\nu(s)$ is determined from eq. (60), which is analogous to eqs. (30), (31), (47) and (48) of the previous sections.

The determination of B_0 proceeds as in the previous sections: we equate to zero the coefficients of k^1 in eqs. (56), (57) and obtain

$$B_{1r} = -a^2/r^2\,\sigma' h' B_0 \text{ at } r = a + h \tag{62a}$$

$$\partial/\partial r\ (B_{1r} J^2) = a^2/r^2\,(2\sigma' B_0' + \sigma'' B_0 + \sigma' B_0\, d/ds) J^2\ . \tag{62b}$$

We can integrate eq. (62b) between $r = 0$ and $r = a + h$, noting that $J_\nu(z) \sim (z/2)^\nu/\Gamma(\nu+1)$ as $z \to 0$. Thus we get from eqs. (62a, 62b) a linear first order ordinary differential equation for B_0, which gives $B_0(s)$ in terms of known functions. Equating to zero the coefficients of k^0, k^{-1} etc. yields recursive ordinary differential equations for $B_1, B_2, \ldots, B_m, \ldots$, in terms of $B_0, B_1, \ldots, B_{m-1}, \ldots$ as before.

Matkowsky [10] has considered some problems of "thin domains" (like our waveguide region) in an even more general way: by assuming the guiding region to be symmetrical about some general arbitrary surface (rather than plane $Y=0$ or the circle $R=A$) he found asymptotic expansions in some arbitrary non-symmetrical regions. Added in proof: we have succeded to generalize our method to non-symmetrical regions as well. This, and more results, will be published soon.

6. Conclusion

A systematic approach to the asymptotic solution of a class of linear partial differential equations in an unbounded space ("radiation" problems) has been to assume a solution of the form [1]:

$$U(X) \sim \exp[ik\sigma(X)] \sum_{m=0}^{\infty} A_m(X)(ik)^{-m}$$

where k is a large parameter that appears in the equations and (possibly) the boundary conditions. Substitution of this assumed solution in the equation and the boundary conditions yields a first order partial differential equation for σ (the eiconal equation), and a recursive system of ordinary differential equations for $A_m(m=0, 1, 2\ldots)$ along the rays, which are the characteristic curves of the eiconal equation.

We have shown that it is possible to reduce some problems of "guided waves", which are eigenvalue problems, to radiation problems in a space that has one dimension less than the original bounded space. This is achieved by assuming a solution of the form

$$u(X) \sim \exp[ik\sigma(X)] f[\gamma(X)] \sum_{m=0}^{\infty} A_m(X)(ik)^{-m} \tag{63}$$

where f is chosen in accordance with the geometry of the problem (an exponential, trigonometric or cylinder function in our examples). A certain additional restriction is imposed on the asymptotic expansion as well (see eqs. (13) and (58)). Substituting (65) in the equations and the boundary conditions yields a system of equations for $\sigma(X)$ and $\gamma(X)$ (for example: eqs. (14) and (15), eqs. (29) and (30) or eqs. (60) and (61a)). Simultaneous solution of these equations yields the spectrum of the problem and determines the ray trajectories for each mode (if there are more than one). In addition, we get again recursive systems of ordinary differential equations for the expansion coefficients A_m along the rays.

The method devised here could be useful in analysing a variety of non-homogeneous waveguide problems, such as the propagation of elastic disturbances in the layers of the earth, propagation of electromagnetic signals in the ionosphere or in the "earth-ionosphere waveguide", propagation of under water sound etc.

Problems of scattering and reflection from discontinuities or obstacles in uniform waveguides have been analysed in terms of the modes of the unperturbed structure. Since we have shown how to calculate the "slowly varying modes" of some non-uniform structures, it seems quite possible that approximate solutions of corresponding scattering and reflection problems in those non-uniform structures can be found in a similar way.

Our method is of course subject to the same restrictions as the above mentioned method for unbounded spaces. We may assume however, that more sophisticated methods (such as the "uniform asymptotic" method [8], [9], [11] can be similarly extended to waveguide problems, and may overcome some of the shortcomings of our method.

After completion of this work the author was made aware by Professor J. B. Keller that some similar results have been derived independently in a somewhat different way by Professor Keller and his co-workers in [2] (chapters X and XI) and in [10]. However, the approach and the scope of this paper seem sufficiently different to justify its separate publication.

The author wishes to express his gratitude to Professor Keller for his interest and useful comments.

Appendix

We shall summarize briefly the assumptions and results of the ray theory as needed for this paper. Our summary follows reference [1]. A short summary of this method is also given in [8].

Consider a function $u(\boldsymbol{x})$, $(\boldsymbol{x}=(x_1, x_2, x_3))$ that satisfies the equation

$$\nabla^2 u + k^2 n^2(\boldsymbol{x}) u = 0 \tag{A.1}$$

in an unbounded space (subject to a radiation condition).

We assume

$$u(\boldsymbol{x}) = A(\boldsymbol{x}, k) e^{ik\sigma(\boldsymbol{x})} \tag{A.2}$$

where A has the asymptotic expansion

$$A(\boldsymbol{x}, k) \sim \sum_{m=0}^{\infty} A_m(\boldsymbol{x})(ik)^{-m} . \tag{A.3}$$

Substituting (A.2) and (A.3) into (A.1) and equating separately to zero the coefficients of each power of k, yields first of all the eiconal equation,

$$(\nabla\sigma)^2 = n^2 \tag{A.4}$$

and the following recursive system for A_m,

$$2\nabla\sigma \cdot \nabla A_0 + A_0 \nabla^2 \sigma = \nabla \cdot (A_0^2 \nabla S)/A_0 = 0 \tag{A.5}$$

$$2\nabla\sigma \cdot \nabla A_m + A_m \nabla^2 \sigma = -\nabla^2 A_{m-1} \qquad m = 1, 2, \ldots \tag{A.6}$$

It can be shown that the rays, (or the characteristics) of eq. (A.4), which are perpendicular to the wavefronts $\sigma = \text{const}$, are solutions of the ordinary differential equations

$$n \frac{d}{ds}\left(n \frac{dx_j}{ds}\right) = \frac{\partial}{\partial x_j}\left(\frac{n^2}{2}\right), \qquad (j = 1, 2, 3) \tag{A.7}$$

where s is an arclength parameter along the curves $\boldsymbol{x} = \boldsymbol{x}(s)$. The phase function σ becomes

$$\sigma(s) = \sigma(s_0) + \int_{s_0}^{s} n[\boldsymbol{x}(s')] \, ds' . \tag{A.8}$$

The vector $\nabla\sigma$ is tangent to the ray, and $|\nabla\sigma| = n$. Thus

$$\nabla\sigma = n \frac{d\boldsymbol{x}}{ds}$$

$$\nabla\sigma \cdot \nabla = n \sum_{i=1}^{3} \frac{dx_i}{ds} \frac{\partial}{\partial x_i} = n \frac{d}{ds} . \tag{A.9}$$

Applying Gauss' theorem to eq. (A.5) over the volume of an infinitesimal "tube of rays" that extends from s_0 to s, yields

$$(A_0^2 n \delta a)_s = (A_0^2 n \delta a)_{s_0} \tag{A.10}$$

where δa is the normal cross-section of the tube.

Applying Gauss' theorem to $\nabla^2\sigma$ over such a tube of rays, and letting $s \to s_0$, yields

$$\frac{\nabla^2 \sigma}{n} = \frac{d}{ds} \ln(n \delta a) . \tag{A.11}$$

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Journal of Engineering Math., Vol. 4 (1970) 261–271